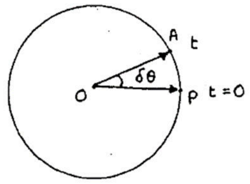
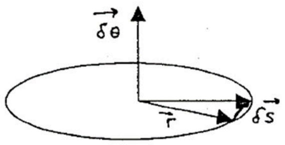
CIRCULAR MOTION

Angular Displacement: is defined as the angle described by the radius vector in a given time at the centre of circle.





When a particle performs circular motion, moves from A to B in short time δt, describes an arc of length δs, then the angle traced by the radius vector at the centre of the circle is $\delta\theta = \frac{\delta s}{r}$.

Instantaneous angular displacement δθ is a vector & its direction is given by right hand thumb rule or right handed screw rule.

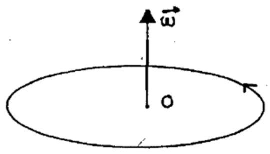
$\vec{\delta s} = \vec{\delta\theta} \times \vec{r}$

Angular Velocity (ω): of a particle performing circular motion is defined as the time rate of change of limiting angular displacement.

Instantaneous angular velocity $\vec{\omega} = \lim_{\delta t \to 0} \frac{\vec{\delta\theta}}{\delta t} = \frac{\vec{d\theta}}{dt}$

Finite angular velocity, $\omega = \frac{\theta}{t}$

S.I. Unit: rad/s Dimensions: $[M^0 L^0 T^{-1}]$



It is a vector quantity and the direction is given by right hand rule.

Angular acceleration (α): is defined as the time rate of change of angular velocity.

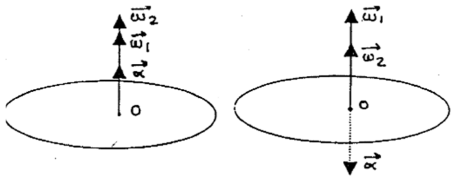
Average angular acceleration $\vec{\alpha} = \frac{\vec{\delta\omega}}{\delta t}$

Instantaneous angular acceleration is defined as the limiting rate of change of angular velocity

$\vec{\alpha} = \lim_{\delta t \to 0} \frac{\vec{\delta\omega}}{\delta t} = \frac{\vec{d\omega}}{dt}$

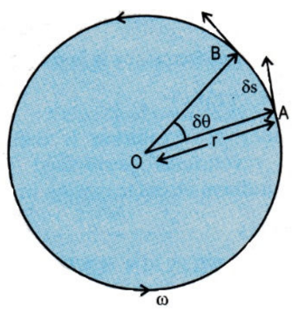
S.I. Unit: rad/s²

Dimensions: $[M^0 L^0 T^{-2}]$



It is vector quantity whose direction is given by right hand rule (positive, same direction as ω; negative, opposite direction to ω]

Relationship between Linear Velocity and Angular Velocity



Consider a particle revolving in anticlockwise sense along the circumference of a circle of radius r. Let δt be the time required by the particle performing circular motion to move from A to B. Let δs be the distance traveled in this time δt.

$\delta s = \delta\theta \times r$

Dividing with δt, $\frac{\vec{\delta s}}{\delta t} = \frac{\vec{\delta\theta}}{\delta t} \times \vec{r}$

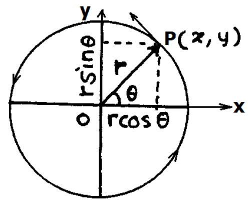
$\lim_{\delta t \to 0} \frac{\vec{\delta s}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{\delta\theta}}{\delta t} \times \vec{r}$

$\frac{\vec{ds}}{dt} = \frac{\vec{d\theta}}{dt} \times \vec{r}$ Therefore, $\vec{v} = \vec{\omega} \times \vec{r}$

In magnitude form v=rω

Uniform Circular Motion (UCM): is defined as the motion of a particle along the circumference of a circle with constant speed.

Radial Acceleration :



Consider a particle performing UCM along the circumference of a circle of radius r in anticlockwise sense. Let P(x,y) be the position of the particle at any instant t. Let θ be the angular displacement at P. The position vector r at any instant t is given as

$\vec{r} = x\,\hat{i} + y\,\hat{j} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} = r\cos\omega t\,\hat{i} + r\sin\omega t\,\hat{j}$

Instantaneous velocity, $\vec{v} = \frac{\vec{dr}}{dt} = \frac{d}{dt}(r\cos\omega t\,\hat{i} + r\sin\omega t\,\hat{j})$

Thus, $\vec{v} = -\omega r\sin\omega t\,\hat{i} + \omega r\cos\omega t\,\hat{j}$

Instantaneous linear accereration, $\vec{a} = \frac{dv}{dt} = \frac{d}{dt}(-\omega r\sin\omega t\,\hat{i} + \omega r\cos\omega t\,\hat{j})$

$\vec{a} = -\hat{i}\,r\,\omega^2\cos\omega t - \hat{j}\,r\,\omega^2\sin\omega t = -\omega^2(\hat{i}\,r\cos\omega t + \hat{j}\,r\sin\omega t) = -\omega^2\vec{r}$

Negative sign indicates that acceleration of the particle performing UCM is oppositely directed to radius vector (i.e. directed inwards. Hence called radial / centripetal acceleration)

Magnitude $a = \omega^2 r = \frac{v^2}{r} = v\,\omega$

Relation between angular acceleration and linear acceleration :

$a = \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt} = \frac{d}{dt}(r\,\omega) = r\frac{d\omega}{dt} + \omega\frac{dr}{dt} = r\frac{d\omega}{dt} + 0$ (since r =constant)

$a = r\,\alpha$

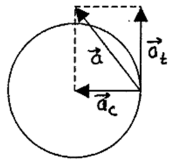
Resultant Acceleration of a particle performing circular Motion :

$\vec{v} = \vec{\omega} \times \vec{r}$

Differentiating, $\frac{\vec{dv}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$

$\frac{\vec{dv}}{dt} = \frac{\vec{d\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{\vec{dr}}{dt}$

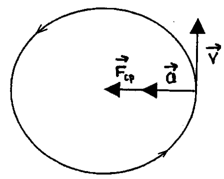
$\frac{\vec{dv}}{dt} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$



$\vec{a} = \vec{a_T} + \vec{a_r}$ Magnitude $a = \sqrt{a_T^2 + a_r^2}$

NOTE : In UCM, ω = const, α = 0, a_T=0 therefore a = a_r

Centripetal Force :



In UCM, a particle experiences centripetal acceleration which is given by $a = -\omega^2 r$

According to Newton's second law of motion, this acceleration must be produced by force acting in the same direction.

Centripetal force is force acting on particle performing circular motion, which is along radius of circle and directed towards the centre of circle.

$F_{cp} = ma = m\,v\,\omega = \frac{m\,v^2}{r} = m\,r\,\omega^2$

In vector notation, $F_{cp} = -\,m\,\omega^2\,\vec{r}$

v : linear speed of particle performing UCM

r : radius of circle; $\vec{r}$: radius vector

ω : angular speed of particle performing circular motion

S.I. Unit: Newton (N)

Features:

>> Real Force

>> Necessary force for maintaining circular motion

>> Direction is different at different points

>> acts along the radius of circle and directed towards the centre

>> Does no work (Since no displacement in direction of force)

Examples:

>> Object tied at the end of a string and whirled in a horizontal circle, the centripetal force is provided by tension in the string

>> Car traveling round a circular track with uniform speed, the centripetal force is provided by the friction between the tyres and the road

>> The electron revolving around the nucleus, the necessary centripetal force is provided by the electrostatic force of attraction between the electron and the nucleus

>> Moon revolves around the Earth, the necessary centripetal force is provided by the gravitational force of attraction between the two.

Centrifugal Force: is a pseudo force in UCM, which acts along the radius and directed away from the centre of circle.

Magnitude of centrifugal force = $m v \omega = \frac{m v^2}{r} = m r \omega^2$

Example:

>> Car in motion takes a sudden left, passengers experience an outward push to the right due to centrifugal force

>> A bucket full of water rotated in a vertical circle at a particular speed, the water does not fall, because, weight of water is balanced by the centrifugal force

>> A coin placed on a turn table moves away as the turn table is rotated at higher speeds.

Centripetal force	Centrifugal force
Real force	Pseudo Force
Force due to interaction between two bodies	Not due to interaction between two bodies
Directed radially inwards	Directed radially outwards
Obeys Newton's laws	Does not obey Newton's laws
Arises in an inertial frame of reference	Arises in a non-inertial frame of reference

Maximum speed of a car along a horizontal curved road:

Consider a car moving along a curved horizontal road

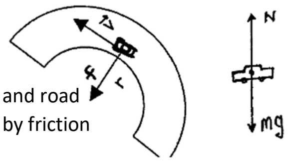
m: mass of the vehicle

r : radius of the road

v: maximum speed of the vehicle

μ : coefficient of friction between the tyres and road

The necessary centripetal force is provided by friction



$\frac{mv^2}{r} = \mu N$; But N = mg

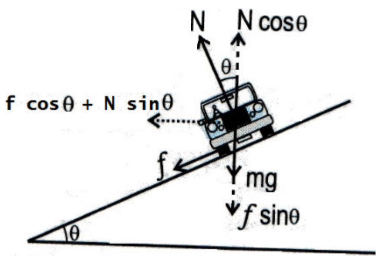
Thus, $\frac{mv^2}{r} = \mu mg$. Hence, $v = \sqrt{\mu rg}$

NOTE: This is the maximum speed for safe turning and is independent of the mass of the vehicle.

Banking of Roads:

Since force of friction is unreliable and it changes with the condition of the tyres, road, weather, etc. Hence banking of roads is the best remedy for vehicle traveling at high speeds along curved road.

The process of raising the outer edge of the road over the inner edge through certain angle is known as banking of road. The angle made by the surface of the road with the horizontal surface of the road is called angle of banking.



Consider a vehicle of mass m traveling with maximum safe speed v. Let θ be the angle of banking and f the force of friction. After resolving the Normal reaction and force of friction we get,

N cosθ = mg + f sinθ .

Thus, mg = N cosθ – f sinθ .....(i)

And the necessary centripetal force is given by

$\frac{mv^2}{r}$ = N sinθ + f cosθ .... (ii)

(ii) divided by (i) gives, $\frac{v^2}{rg} = \frac{N\sin\theta + f\cos\theta}{N\cos\theta - f\sin\theta}$

Put $f = \mu_s N$ (maximum value of friction) we get,

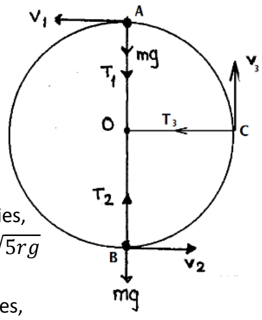
$v_{max} = \sqrt{rg\left[\frac{\mu+\tan\theta}{1-\mu\tan\theta}\right]}$

Case 1: Horizontal road, θ=0°, $V_{max} = \sqrt{\mu rg}$

Case 2: No friction μ = 0, optimum speed = $v_o = \sqrt{rg\tan\theta}$

$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$, is independent of mass of vehicle

Vertical Circular Motion:



>>Velocity at highest point A

At A, $T_1 + mg = \frac{mv_1^2}{r}$

To just complete a circle, v_1 should be minimum. Thus, $T_1=0$

thus, $v_1 = \sqrt{rg}$

>>Velocity at lowest point B

$TE_A = TE_B$, i.e. $KE_A + PE_A = KE_B + PE_B$ which implies,

$\frac{1}{2} m(\sqrt{rg})^2 + mg(2r) = \frac{1}{2} mv_2^2 + 0$, giving $v_2 = \sqrt{5rg}$

>>Velocity midway at point C

$TE_B = TE_C$, i.e. $KE_B + PE_B = KE_C + PE_C$ which implies,

$\frac{1}{2} m(\sqrt{5rg})^2 + mg(0) = \frac{1}{2} mv_3^2 + mgr$, giving $v_3 = \sqrt{3rg}$

Total Energy at A	Total Energy at B	Total Energy at C
$KE = \frac{1}{2} m v_1^2 = \frac{1}{2} m(\sqrt{rg})^2 = \frac{mgr}{2}$	$KE = \frac{1}{2} m v_1^2 = \frac{1}{2} m(\sqrt{5rg})^2 = \frac{5mgr}{2}$	$KE = \frac{1}{2} m v_1^2 = \frac{1}{2} m(\sqrt{3rg})^2 = \frac{3mgr}{2}$
PE = mg (2r)	PE = 0	PE = mg (r)
$TE = \frac{5mgr}{2}$	$TE = \frac{5mgr}{2}$	$TE = \frac{5mgr}{2}$
Total energy is conserved		

>>Difference between the tension at lowest and highest points is 6mg

At A, $T_1 = \frac{mv_1^2}{r} - mg$, At B, $T_2 = \frac{mv_2^2}{r} + mg$

Thus, $T_2 - T_1 = \frac{mv_2^2}{r} + mg - \frac{mv_1^2}{r} + mg = 2mg + \frac{m}{r}(v_2^2 - v_1^2)$

Using $v^2 - u^2 = 2gh$ (-or- loss in KE = gain in PE) we get

$T_2 - T_1 = 2mg + \frac{m}{r}(2g.2r) = 2mg + 4mg = 6mg$

Conical Pendulum:

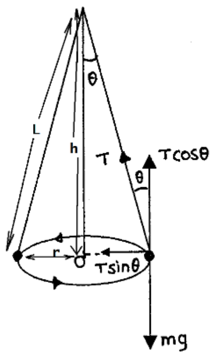
A conical pendulum is a simple pendulum, which is given such a motion that bob describes a horizontal circle and the string describes a cone.

After resolving T into its components we get,

Tcosθ=mg

and the centripetal force is provided by

$T\sin\theta = \frac{mv^2}{r}$



Dividing the two, we get, $v = \sqrt{rg\tan\theta}$

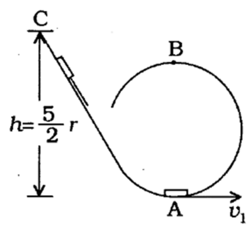
Time Period $T = \frac{2\pi r}{v} = \frac{2\pi r}{\sqrt{rg\tan\theta}}$

$T = 2\pi\sqrt{\frac{r}{g\tan\theta}} = 2\pi\sqrt{\frac{r}{g\left(\frac{r}{h}\right)}}$ [Since $\tan\theta = \frac{r}{h}$]

Thus, $T = 2\pi\sqrt{\frac{h}{g}} = 2\pi\sqrt{\frac{L\cos\theta}{g}}$ [Since h = Lcosθ]

If θ is small, cosθ ~ θ and $T = 2\pi\sqrt{\frac{l}{g}}$

Looping the Loop



$TE_C = mgh$, $TE_A = \frac{1}{2} mv_A^2$

$TE_C = TE_A$ gives $V_A = \sqrt{2gh}$

To just complete the circle $V_A = \sqrt{5rg}$

Thus, $\sqrt{2gh} = \sqrt{5rg}$. Thus, $h = 5r/2$

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